ON THE TESTS OF A COMMON MEAN OF NORMAL POPULATIONS WITH UNKNOWN VARIANCES

By M. B. Golhar

*Institute of Agricultural Research Statistics (I.C.A.R.), New Delhi**

1. Introduction

We are given two normal populations with a common but unknown mean, say μ, and unknown variances, say σ_1^2 and σ_2^2, about which no information is available about their equality.

We wish to test the hypothesis that μ is equal to some preassigned value, say μ_0.

Yates (1939) has suggested a method by which the Behrens-Fisher distribution can be used for this test. But this test gives the probability of rejection, when the hypothesis is true, greater than α, the nominal significance level. This probability for some values of the parameters have been calculated by James (1959).

James (1956) has suggested an asymptotic method based on Welch's approach (1947) of the Behrens-Fisher problem. James (1959) has discussed his method and Yate's method in his paper.

Scheffe's approach (1943) of the Behrens-Fisher problem can also be used for finding a t-statistic for the present problem. He has considered a non-symmetric linear function of the sample observations which would give the test criterion. This statistic depends upon the order of the sample observations so that the randomisation of the sample observations is necessary. As the different orderings of the sample values give different values of the statistic, the inference drawn from the data by this test seems to have a certain random element in it.

However, we shall proceed to find a similar t-statistic for the present problem.

---

* Now with Psychometric Unit, National Council of Educational Research and Training, Delhi-1.

2.1. *Test Criterion*

Let $x_1 \cdots x_m$, and $y_1 \cdots y_n$ be the two samples of sizes m and n from the two normal populations $N(\mu, \sigma_1^2)$ and $N(\mu, \sigma_2^2)$ respectively, where μ and σ_i^2 are unknown.

With no loss of generality we assume $n \geqslant m$. Let S_1^2 and S_2^2 be the estimates of σ_1^2 and σ_2^2 respectively. We wish to test the hypothesis H_0, where $H_0 : \mu = \mu_0$, by the use of a t-statistic.

We define linear functions of the sample variables x_j and y_j by

$$z_i = \sum_{j=1}^{m} a_{ij} x_j + \sum_{j=1}^{n} \beta_{ij} y_j, \quad (i = 1, 2, \cdots f); \tag{1}$$

the coefficients a_{ij} and β_{ij} being subject to the conditions that the random variables $z_i \cdots (i = 1 \cdots f)$ are independently and normally distributed with the same mean μ and a common variance, say, σ^2.

The following conditions are obtained:—

$$(a) \quad \sum_{j=1}^{m} a_{ij} + \sum_{j=1}^{n} \beta_{ij} = 1, \quad (i = 1, 2, \cdots f);$$

$$(b) \quad \sum_{j=1}^{m} a_{ij} a_{kj} = c_1^2 \delta_{ik}, \quad (i, k = 1, 2, \cdots f);$$

and

$$\sum_{j=1}^{n} \beta_{ij} \beta_{kj} = c_2^2 \delta_{ik};$$

where

$$\delta_{ik} = 0, \text{ if } i \neq k,$$
$$= 1, \text{ if } i = k;$$

where

$$\sigma^2 = c_1^2 \sigma_1^2 + c_2^2 \sigma_2^2,$$

and

$$(i, k = 1, 2, \cdots f).$$

Let $\bar{z}$ be the mean of z_i's.

We may now define a t-statistic from these variables. The critical region in f-dimensional sample space is, for the level of significance α, W:

$$\frac{|\bar{z}-\mu_0|}{\sqrt{\dfrac{\sum_{i=1}^{f}(z_i-\bar{z})^2}{f(f-1)}}}=t\geqslant t_\alpha;$$

the limit of significance t_α, drawing from t-tables for $(f-1)$ degrees of freedom.

It may be seen that to achieve the maximum power for the test, we should find a variable of the type Z_i, whose variance is minimum value that can be associated with f concerned. The optimum value of f is equal to m, which was shown by Scheffe (1943). So we seek the solution with minimum c_1^2 and c_2^2.

2.2. *Solution to the* t-*statistic*

Let $A(m\times m)$ and $B(m\times n)$ be two matrices, where $A(m\times m) = (\alpha_{ij})$, and $B(m\times n)=(\beta_{ij})$.

Let $(A/B)(m\times\overline{m+n})$ be the matrix having its i-th row as $(\alpha_{i1}, \alpha_{i2}, \cdots \alpha_{im}, \beta_{i1}, \beta_{i2}\cdots\beta_{in})$. We denote $(A/B)=(\gamma_{ij})\ (m\times\overline{m+n})$.

Then the following conditions (c) can be obtained from the conditions (a) and (b).

$$(c)\quad \sum_{j=1}^{m+n}\gamma_{ij}=1,$$

and

$$\sum_{j=1}^{m+n}\gamma_{ij}\gamma_{kj}=(c_1^2+c_2^2)\,\delta_{ik};\quad (i,k=1,2,\cdots m).$$

As shown by Scheffe (1943), the minimum value of $c_1^2+c_2^2$ is $m/(m+n)$.

Now add a set of $(n-m)$ rows to m rows of B to make a set of n orthogonal vectors with its norm c_2^2, denoting additional elements by β_{ij}'s.

Let

$$\sum_{j=1}^{m}\alpha_{ij}=a_i,\qquad (i=1,2,\cdots m);$$

and

$$\sum_{j=1}^{n} \beta_{ij} = b_i, \quad (i = 1, 2, \cdots n); \tag{2}$$

where

$$a_i + b_i = 1, \quad \text{for} \quad (i = 1, 2, \cdots m).$$

Let α_i and β_i be the rows of A and B respectively, and $\phi\,(1 \times m)$ and $\psi\,(1 \times n) = (1, 1, \cdots 1)$ the unit row vectors. Then (2) can be written as

$$\alpha_i \phi' = a_i, \quad (i = 1, 2, \cdots m);$$

and

$$\beta_i \psi' = b_i, \quad (i = 1, 2, \cdots n); \tag{3}$$

where ϕ' and ψ' are unit column vectors.

Also the conditions (a) and (b) can be written as:

$$a_i + b_i = 1, \quad (i = 1, 2, \cdots m);$$

$$\alpha_i \alpha_j' = c_1^2 \delta_{ij}, \quad (i, j = 1, 2, \cdots m);$$

and

$$\beta_i \beta_j' = c_2^2 \delta_{ij}, \quad (i, j = 1, 2, \cdots n). \tag{4}$$

As $\beta_1 \cdots \beta_n$ form an orthogonal basis in n-space, we have $\psi = \sum_{k=1}^{n} g_k \beta_k$, where g_k's are scalars. Substituting the value of ψ in (3), we get $b_i = \beta_i \psi' = \sum_{k=1}^{n} \beta_i g_k \beta_k' = g_i c_2^2$, $(i = 1, 2, \cdots n)$.

So

$$g_i = \frac{b_i}{c_2^2}, \quad (i = 1, 2, \cdots n).$$

Also

$$\psi\psi' = \sum_{k=1}^{n} g_k^2 \beta_k \beta_k' = \sum_{i=1}^{n} \frac{b_i^2}{c_2^2}.$$

So we get the value of c_2^2, where

$$c_2^2 = \frac{\sum_{i=1}^{n} b_i^2}{n}.$$

Similarly,

$$c_1^2 = \frac{\sum_{i=1}^{m} a_i^2}{m}.$$

Without violating the conditions we may put the restrictions $a_1 = a_2 = \cdots = am = a$, say.

Then we get $c_1^2 = a^2$, and the minimum value of

$$c_2^2 = \frac{m}{n}(1-a)^2.$$

It can be proved that this minimum value can be attained by c_2^2, (*see* Scheffe, 1943). Combining these values with

$$c_1^2 + c_2^2 = \frac{m}{m+n},$$

we get

$$a = \frac{m}{(m+n)},$$

and

$$a_{ij} = \frac{m}{m+n}\delta_{ij}, \quad (i, j = 1, 2, \cdots m);$$

and

$$\beta_{ij} = \frac{\sqrt{mn}}{m+n}\delta_{ij} - \frac{1}{m+n}\left(\sqrt{\frac{n}{m}} - 1\right), \text{ for } j \leqslant m,$$

$$= \frac{1}{m+n}, \text{ for } n \geqslant j > m; \text{ and } (i = 1, 2, \cdots m).$$

Substituting the values of a_{ij} and β_{ij} in (1), and simplifying, we get

$$z_i = \frac{m}{m+n}x_i + \frac{n\bar{y}}{m+n} + \frac{\sqrt{mn}}{m+n}y_i - \frac{\sqrt{\frac{n}{m}}}{m+n}\sum_{j=1}^{m} y_j,$$

$$(i = 1, 2, \cdots m),$$

and

$$\bar{z} = \frac{m\bar{x} + n\bar{y}}{m + n}.$$

If z_i is replaced by the new variable u_i, where

$$u_i = \frac{\sqrt{m}}{m + n}(\sqrt{m}\, x_i + \sqrt{n}\, y_i), \quad (i = 1 \cdots m);$$

and

$$\bar{u} = \frac{\sqrt{m}}{m + n}\left(\sqrt{m}\,\bar{x} + \sqrt{n}\,\frac{\sum_{i}^{m} y_j}{m}\right);$$

we get

$$\sum_{i=1}^{m} (z_i - \bar{z})^2 = \sum_{i=1}^{m} (u_i - \bar{u})^2 = Q, \text{ say.}$$

Then we can write the t-statistic as

$$t = \frac{\bar{z} - \mu_0}{\sqrt{\dfrac{Q}{m(m-1)}}},$$

which is distributed in t-distribution with $(m - 1)$ degrees of freedom if the hypothesis is true.

2.3. *Remarks on* t-*Criterion*

Scheffe's approach utilises the whole data which is an improvement over the method previously suggested by Bartlett. Still this approach is, perhaps, slightly less efficient than Welch's approach, in the sense that the power of Scheffe's test is slightly less than the power of Welch's test. The power of Welch's test was calculated by the author (1964) for some particular values of the parameters. This slight loss of efficiency may be explained as the effect of the fact that Scheffe takes the first kind of error exactly equal to α, the nominal significance level; and Welch takes a slight freedom at this point for an asymptotic solution by approximating it nearly equal to α.

The same remarks can also be made in the present case.

Dr. M. N. Ghosh has pointed out that the same t-statistic can also be obtained from the results of his paper (1961). It may be remarked that a large class of problems can be solved in a similar way.

3.1. *Combination of Two Independent Tests*

The problem can also be tackled from a different point of view. The two different independent tests of hypothesis, namely of $\mu_1 = \mu_0$ and $\mu_2 = \mu_0$, can be combined into a single test which weights one test relative to the other, μ_1 and μ_2 being the means of two normal populations with the unknown variances σ_1^2 and σ_2^2 respectively. General discussion of this method may be found in Birhaurn (1954), Good (1955) and Zelen and Joel (1959), etc.

Without loss of generality we shall assume $\mu_0 = 0$.

Consider two independent tests of hypothesis $\mu_1 = 0$, and $\mu_2 = 0$, for two different population means, based on student's t-statistic.

Let

$$t_1 = \frac{\bar{x}_1 - \mu_1}{\frac{S_1}{\sqrt{m}}}, \quad \text{and} \quad t_2 = \frac{\bar{x}_2 - \mu_2}{\frac{S_2}{\sqrt{n}}}.$$

Under the hypothesis H_0: $\mu_1 = 0$, and $\mu_2 = 0$, t_1 and t_2 follow student's t-distributions with $(m - 1)$, and $(n - 1)$ degrees of freedom respectively. If $\mu_1 \neq 0$ and $\mu_2 \neq 0$; then t_j $(j = 1, 2)$ follows non-central t-distribution with the parameter δ_j, where

$$\delta_1 = \frac{m\mu_1^2}{\sigma_1^2}, \quad \text{and} \quad \delta_2 = \frac{n\mu_2^2}{\sigma_2^2}.$$

For the purpose of combining these two independent tests, we consider the integral transformations $P_j = \text{Prob. } [t \geqslant t_j / H_0]$, $(j = 1, 2)$; which is the probability of student's t exceeding the calculated t_j if the null hypothesis is true. Then the critical region of the combined test will be given by W_0, where $W_0 : [P_1 P_2^\theta \leqslant c_\alpha]$; c_α being a constant depending on α, the level of significance; and θ a weighting factor, $(0 \leqslant \theta \leqslant 1)$.

The problem of the choice of θ has been considered by Zelen and Joel (1959), and it is claimed that the choice of $\theta = (\delta_2/\delta_1)$ results in minimum type II error over the wide range of values for the parameters. As σ_1^2/σ_2^2 is not known, its estimate may be used to find the value of θ. But this recommendation, of course, may not give exactly the maximum power. It is shown that a little error in θ does not change the power appreciably because of the flatness of the power curve at that point. The values of c_α have been given by Zelen and Joel (1959), and also a method is suggested to calculate the power-function of the test.

3.2. *Power-function of the Combined Test*

The main formulæ are given below to calculate the power-function of the combined test. The details may be seen in Zelen and Joel (1959).

Power-function of the combined test $= \int\int_{w_0} d\pi_1 d\pi_2$, where, for the significance level α, w_0 being the region of integration given by $[P_1 P_2^\theta \leqslant c_\alpha]$, in (P_1, P_2) space, which can be transformed in (π_1, π_2) space by the following transformations:

$$\pi_j = \int_{x_j}^{1} p(x_j/\delta_j)\, dx_j,$$

and

$$P_j = \int_{x_j}^{1} p(x_j/0)\, dx_j, \qquad (j = 1, 2); \tag{5}$$

where

$$p(x_j | \delta_j) = e^{-(\delta_j/2)} \sum_{r=0}^{\infty} \frac{1}{r!} \frac{\left(\frac{\delta_j}{2}\right)^r x_j^{r-\frac{1}{2}} (1 - x_j)^{(f_j/2)-1}}{B\left(r + \frac{1}{2}, \frac{f_j}{2}\right)},$$

$$(0 \leqslant x_j \leqslant 1);$$

and

$$f_1 = m - 1,\ f_2 = n - 1. \tag{6}$$

(6) is obtained by the transformation

$$x_j = \frac{\frac{t_j^2}{f_j}}{1 + \frac{t_j^2}{f_j}}$$

from non-central t-distribution of t_j.

Using Patnaik's approximation (1949) to the non-central t-distribution we can write (5) as:

$$\pi_j = I_{1-x_j'}\left(\frac{f_j}{2}, \frac{l}{2}\right),$$

and

$$p_j = I_{1-x_j}\left(\frac{f_j}{2}, \frac{1}{2}\right), \quad (j = 1, 2);$$

where

$$l = \frac{(1+\delta_j)^2}{1+2\delta_j}, \quad \text{and} \quad x_j' = \frac{(1+\delta_j)\,x_j}{1+2\delta_j - \delta_j x_j};$$

and $I_y(p, q)$ denoting

$$\int_0^y \frac{x^{p-1}(1-x)^{q-1}}{B(p, q)}\,dx;$$

the values of which can be taken from the tables of incomplete beta-function by K, Pearson.

The power-function of the combined test computed from the above formulæ for some particular values of the parameters is given in Table I, for the significance level $\alpha = \cdot 05$, θ being the constant which is taken as equal to the estimate of δ_2/δ_1.

TABLE I

f_1	f_2	δ_1	δ_2	θ	Power
10	10	2·4142	2·4142	1	·431
10	10	2·4142	2·4142	·1	·302
20	10	2·4112	2·4142	1	·438
20	10	2·4142	2·4142	·1	·320
10	20	2·4142	2·4142	1	·437
10	20	2·4142	2·4142	·1	·305
10	..	2·4142	0	1	·205
10	..	2·4142	0	·1	·278
20	..	2·4142	0	1	·225
20	..	2·4142	0	·1	·295
10	10	2·4142	2·4142	·5	·391
20	10	2·4142	2·4142	·5	·408
10	20	2·4142	2·4142	·5	·402

The values of the power-function are approximate at the second decimal place by at most one unit because of the use of Patnaik's approximation.

The power of the previous t-test for the corresponding parameters is given in Table II. The values of δ_1, δ_2, f_1 and f_2 decide the value of the non-centrality parameter ρ on which the power depends, where

$$\rho = \frac{\mu(m+n)}{\sqrt{m\sigma_1^2 + n\sigma_2^2}} = \frac{m+n}{\sqrt{\dfrac{m^2}{\delta_1} + \dfrac{n^2}{\delta_2}}}.$$

The values of the power-function are taken from the tables given by Neyman and Tokarska (1936).

TABLE II

f_1	f_2	$\delta_1 = \delta_2$	ρ	Power
10	10	2·4142	2·197	·65
10	20	2·4142	2·097	·62
20	10	2·4142	2·097	·62

The difference between the two approaches considered here is : The first test is itself a t-test, while the second one is a combination of two independent t-tests. When the hypothesis is not true, the first t-test still assumes the equality of means in finding its power-function while in the combination of two t-tests the two means need not be equal.

In the cases considered in Tables I and II, it may be seen that the power of the t-test is greater than the power of the combined test.

4. SUMMARY

Scheffe's approach is applied to the problem of testing the common mean of two normal populations having unknown variances, and a t-statistic is obtained. The problem is also tackled by combining two different independent tests and the power-function of this combined test is calculated for some particular values of the parameters, for the comparison with t-test,

5. Acknowledgements

The author is grateful to Dr. M. N. Ghosh for his helpful suggestions in the preparation of this paper.

The author also thanks Dr. V. G. Panse for his interest and encouragement during the course of the work.

6. References

1. Birbaum, A. .. "Combining independant tests of significance," *J. Am. Stat. Assoc.*, 1954, **49**, 559–74.
2. Ghosh, M. N. .. "Linear estimation of parameters in the case of unequal variances," *Cal. Stat. Assoc. Bull.*, 1961, **10**, 117–30.
3. Golhar, M. B. .. "On the comparison of two means from normal populations with unknown variances," *J. Ind. Soc. of Agr. Stat.*, 1964, 16.
4. Good, I. J. .. "On the weighted combination of significance tests," *J.R.S.S.*, 1955, **17 B**, 264–65.
5. James, G. S. .. "On the accuracy of weighted means and ratios," *Biometrika*, 1956, **43**, 304–21.
6. ———— .. "Behrens-Fisher distribution and weighted means," *J.R.S,S.*, 1959, **21 B**, 73–90.
7. Neyman, J. and Tokarska, B. "Errors of the second kind in testing 'Student's' hypothesis," *J. Am. Stat. Assoc.*, 1936, **31**, 318–26.
8. Patanik, P. B. .. "The non-central χ^2 and F-distributions and their applications," *Biometrika*, 1949, **36**, 202–32
9. Pearson, K. .. *Tables of Incomplete Beta-function*, 1934.
10. Scheffe, H. .. "On solutions of the Behrens-Fisher problem based on *t*-distribution," *Ann. of Math. Stat.*, 1943, **14**, 35–44.
11. Welch, B. L. ... "Generalisation of 'Student's' problem when several different population variances are involved," *Biometrika*, 1947, **34**, 28–35.
12. Yates, F. .. "An apparent inconsistency arising from tests of significance based on fiducial distributions of unknown parameters," *Proc. Camb. Phil. Soc.*, 1939, **35**, 579–91.
13. Zelen, M. and Joel, L. S. "Weighted compounding of two independen significance tests," *Ann. of Math. Stat.*, 1959, **30**, 885–95.